Balanced LQG Compensator for Flexible Structures

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Abstract

The analysis of open-loop balanced flexible structures has been extended for closed-loop structures. LQG compensator gains (i.e., the gains of a . controller and of an estimator) are obtained from the solutions of the controller Riccati equation (CARE) and the estimator Riccati equation (FARE). For the balanced compensator the solutions of CARE and FARE are equal and diagonal. Thus, a balanced LQG compensator puts the same effort into control and estimation of the system. An approximate balanced LQG compensator for flexible structures is determined in this paper. Its properties allow one to obtain a reduced-order compensator, which preserves the stability and performance of the full-order compensator. The performance of an LQG compensator depends on the weights of the quadratic performance index and on the variance of the estimator noise. The relationships between weights/variances and characteristic values of the system as well as between weights/variances and plant/estimator pole location are derived in this paper. Thus the weights can be determined in advance to meet the requirements of a closed-loop system.

1. Introduction

Control issues for flexible structures have gained increasing attention, especially in space applications. The growing interest reflects recent efforts to maintain high precision positioning of ever lighter and more flexible structures. This paper contributes to this effort by developing a balanced LQG compensator for flexible structures. There have been many investigations into analysis and design of LQG compensators, and a good insight into the variety of approaches can be obtained from Kwakernaak and Sivan (197?), Maciejowski (1989), Anderson and Moore (1990), and Furuta and Sano (1988). The LQG design procedures yield an optimal compensator. However, an optimal solution is not necessarily a reasonable one, since it is dependent on the weights of the quadratic performance index and on the variance of the estimator noise. Thus, the index weight and filter covariance need to be predetermined in order to obtain a reasonable performance. The relationships between weights/variances and characteristic values of the system, as well as between weights/variances and plant/estimator pole location, are derived in this paper, making possible the design of an optimal compensator that satisfies the requirements.

The controller and estimator gains of an LQG compensator are obtained from the solutions of the controller Riccati equation (CARE) and the estimator Riccati equation (FARE). In the approach presented, the equal and diagonal solution of CARE and FARE is sought. The equal and diagonal solution of CARE and FARE is the balanced LQG solution, and its diagonal entries are the characteristic values of the system (Jonckheere and Silverman, 1983). Jonckheere and Silverman show for a specific case that the balanced LQG solution exists. in this paper the transformation to the balanced LQG representation is derived for the general case, It is also shown that flexible structures in a Moore balanced representation (Moore, 1981; Gawronski and Juang, 1990; Gawronski and William, 1991) are approximately LQG balanced. In the LQG balanced representation a balanced performance is obtained for a controller and an estimator. Thus the action of a highly efficient controller is not deteriorated by poor estimator accuracy, nor on the other hand is it overdetermined by an overperforming estimator.

The LQG balanced representation is used for compensator reduction. The pole mobility index characterizes the importance of the closed loop component of the compensator. The states with small mobility index are truncated, leaving a closed-loop system with a stable reduced compensator.

2. LQG Compensator

In this paper a flexible structure is defined as a controllable and observable linear system with distinct complex conjugate pairs of poles (N poles, N is even), and with small and negative real parts of the poles. In other words, it is a linear system with vibrational properties. In the Moore balanced coordinates it consists of n = N/2 components (Gawronski and Juang, 1990; Gawronski and Williams, 1991, and each component consists of two states.

Let (A, B, C) be a state-space triple of a flexible structure. Its controllability and observability grammians W_c and W_o are positive-definite and satisfy the Lyapunov equations

$$A W_c + W_c A^T + BB^T = 0, \qquad A^T W_o + W_o A + C^T C = 0 \tag{1}$$

The system representation is balanced in the sense of Moore (C. f, , Moore, 1981) if its controllability and observability grammians are diagonal and equal

$$W_c = W_o = \Gamma^2, \quad \Gamma = diag(\gamma_1, \ldots, \gamma_N), \; i = 1, \ldots, N \tag{2}$$

where $\gamma_i > 0$ is the ith Hankel singular value of the system.

Consider a flexible structure with an LQG compensator as in Fig. 1. The noises v and w are uncorrelated, where. v is the process noise with intensity V, and w is measurement noise with intensity W

$$V = E(vv^T), \; W = E(ww^T), \; E(vw^T) = 0, \; E(v) = 0, \; E(w) = 0 \tag{3}$$

where $E(.)$ is an expectation operator. It is assumed that $W = I$ without loss of generality. The task is to determine the controller gain (K_p) and estimator gain (K_e) such that the performance index J

$$J^2 = E\left(\int_0^\infty (x^T Q x + u^T R u) dt \right) \tag{4}$$

is minimal, where A' is a positive definite input weight matrix, and Q a positive semi-definite state weight matrix. It is assumed $R = 1$ without loss of generality.

The minimum of J is obtained for the feedback $u = -K_p x$, where. the gain matrix

$$K_p = B^T S, \tag{5}$$

is obtained from the solution S of the controller Riccati equation (CARE) (Kwakernaak and Sivan, 1972)

$$A^T S + SA - SBB^T S + Q = 0 \tag{6}$$

The optimal estimator gain is given by

$$K_e = PC^T, \tag{7}$$

where 1' is the solution of the estimator Riccati equation (l'Al{li})

$$AP + PA^T - PC^T CP + V = 0. \tag{8}$$

3. Balanced LQG Compensator

The balancing of CARE and FARE equations is considered. Jonckheere and Silverman (1983), and Opdenacker and Jonckheere (1985) have shown that a balanced solution for CARE and FARE equations exists in case of $Q = C^T C$ and $V = BB^T$. Namely, there exists a diagonal positive definite $M = diag(\mu_i)$, $i = 1, \ldots, n$, $\mu_i > 0$, such that

$$S = P = M \tag{9}$$

A state-space representation with the condition (9) satisfied is called an LQG balanced representation, and $\mu_i, i = 1, \ldots, n$ are the characteristic values of (A, B, C).

Consider the transformation T of the state x such that $x = T\bar{x}$, then $\bar{A} = T^{-1}AT$, $\bar{B} = T^{-1}B$, $\bar{C} = CT$; in the new coordinates

$$\bar{S}_c = T^T S_c T,\ \bar{Q}_c = T^T Q_c T,\ \bar{S}_f = T^{-1} S_f T^{-T},\ \bar{Q}_f = T^{-1} Q_f T^{-T} \tag{10}$$

The solution of CARE and FARE is LQG balanced if

$$\bar{S}_c = \bar{S}_f = M,\ M = diag(\mu_1, \mu_2, \ldots, \mu_n),\ \mu_1 \geq \mu_2 \geq \ldots \geq \mu_n > 0 \tag{11}$$

There exists a transformation T, such that CARE and FARE are balanced.

Result 1. The transformation T to the LQG balanced representation is obtained as follows. Decompose S_c and S_f

$$S_c = P_c^T P_c, \quad S_f = P_f P_f^T \tag{12}$$

and form a matrix H

$$H = P_c P_f \tag{13}$$

Find the singular value decomposition of H

$$H = VMU^T \tag{14}$$

then

$$T = P_f U M^{-1/2} = P_c^{-1} V M^{1/2}, \quad T^{-1} = M^{-1/2} V^T P_c = M^{1/2} U^T P_f^{-1} \tag{15}$$

The introduction of the above transformation T to (10) shows that (11) is satisfied.

Now consider weighting matrices of special form, and the corresponding balanced solution.

Result 2. For a folly controllable system, and the weights Q_c and Q_f as follows

$$Q_c = W_c^{-1} B (I + R_c^{-1}) B^T W_c^{-1}, \quad Q_f = W_o^{-1} C^T (I + R_f^{-1}) C W_o^{-1} \tag{16}$$

one obtains CARE and FARE solutions as follows

$$S_c = W_c^{-1}, \quad S_f = W_o^{-1} \tag{17}$$

Proof. Introduction of Eqs. (16) and (17) to CARE gives

$$A^T S_c + S_c A + S_c BB^T S_c = 0 \tag{18}$$

which is the Lyapunov equation (1) for $S_c = W_c^{-1}$. Similar proof can be shown for the FARE solution. □

The weights as in Eq. (16) are for collocated sensors and actuators, and penalize each open-loop balanced state reciprocally to its degree of controllability and observability, trying to make each state of plant and estimator equally influenced by the feedback.

Corollary 1. In the Moore balanced representation $W_c = W_o = \Gamma^2$, thus for weights Q_c, Q_f as in Eq. (16) one obtains an LQG balanced system, with $M = \Gamma^{-2}$.

The matrix Π

$$\Pi = \Gamma_o^2 \Gamma_c^{-2} = diag(\pi_i) = diag(\gamma_{oi}^2 / \gamma_{ci}^2) \tag{19}$$

is the ratio of open- and closed loop Hankel singular values, or the ratio of open- and closed loop state variances excited by the while noise input, Thus Π represents the closed-loop Performance. For weights as in Result 2, and a system in the Moore balanced coordinates one obtains

Corollary 2. In the Moore balanced representation, for weights as in (16)

$$\Pi = 3I \tag{20}$$

Proof. The Lyapunov equation for the closed-loop controllability grammian W_c is as follows

$$(A - BB^T S_c) W_c + W_c (A^T - S_c BB^T) + BB^T = 0 \tag{21a}$$

According, to (17) $S_c = \Gamma_o^{-2}$, and introducing $W_c = \Gamma_o^2 / 3$ to (21a), one obtains

$$A\Gamma_o^2 + \Gamma_o^2 A^T + BB^T = 0 \tag{21b}$$

which shows that $W_c = \Gamma_o^2 / 3$ is a solution of (21a), and consequently that $\Pi = 3I$. □

Result 3. For a folly controllable system, and the weights Q_c and Q_f as follows

$$Q_c = C^T C + W_o B R_c^{-1} B^T W_o, \quad Q_f = BB^T + W_c C^T R_f^{-1} C W_c \tag{22}$$

one obtains CARE and FARE solutions as follows

$$S_c = W_o, \quad S_f = W_c \tag{23}$$

Proof. By introduction of (22) to CARE and FARE equations.

Corollary 3. In the Moore balanced representation $W_c = W_o = \Gamma^2$, thus for weights Q_c, Q_f as in Eq. (22) one obtains an LQG balanced system, with $M = \Gamma^2$.

Define $C_o^T = [C^T\ W_o B R_c^{-1/2}]$, $B_o = [B\ \ W_c\ \ C^T R_f^{-1/2}]$, then the LQG closed-loop system is interpreted as a system with unitary weights and with the auxiliary inputs and outputs as defined by matrices B_o, C_o. For collocated sensors and actuators, and for $R_c = R_f = I$, one obtains $C_o^T = B_o = [I\ W_c] B = [I\ W_o] C^T$ and $Q_c = Q_o$.

4. Approximately Balanced LQG Compensator

In the following sections an approximate equality between two variables is used in the following sense. Two variables x and y are approximately equal ($x \cong y$) if $x = y + \epsilon$, and $\|\epsilon\| / \|y\| \ll 1$.

It will be shown that for flexible structures the balanced representation (in the Moore sense) produces diagonally dominant solutions of CARE and FARE, and in the case of $Q = V$ it produces approximate LQG balanced solutions S and P, such that $S \cong P = M$. In order to prove it, assume a diagonal weight matrix Q

$$Q = diag(q_i I_2), \qquad i = 1, \ldots, n. \tag{24}$$

then the following is true.

Result 4a. There exist $q_i \leq q_{oi}$, where $q_{oi} > 0$, $i = 1, \ldots, n$, such that $S \cong diag(s_i I_2)$ is the solution of (6), where

$$s_i \cong (\beta_{pi} - 1) / 2\gamma_i^2, \quad \beta_{pi}^2 = 1 + 2 q_i \gamma_i^2 / \zeta_i \omega_i \tag{25}$$

Proof is presented in the Appendix.

A similar result is obtained for the FARE equation, namely, for a diagonal V

$$V = diag(v_i I_2), \qquad i = 1,\ldots,n. \tag{26}$$

the following is true:

Result 4b. There exist $v_i \leq v_{oi}$ where $v_{oi} > 0$, $i = 1,\ldots,n$, such that $P = diag(p_i I_2)$ is the solution of (8), where

$$p_i \cong (\beta_{ei}-1)/2\gamma_i^2, \; \beta_{ei}^2 = 1 + 2v_i\gamma_i^2/\zeta_i\omega_i \tag{27}$$

Proof is similar to Resell 2a,

If the i-th diagonal entry of P and the respective entry of S arc equal, say to μ_i, i.e.,

$$p_i = s_i = \mu_i \tag{28}$$

the i-th component is LQG balanced. Additionally, if S and P are equal, as in Eq. (9), where $M = diag(\mu_i)$, $i = 1,\ldots,n$, the system is LQG balanced. If S, P, M arc diagonally dominant, i.e., $V_i + c_{vi} \cong s_i - 1 c_{si} \epsilon_i t_i$ with ϵ_{vi} and ϵ_{si} small ($|\epsilon_{vi}/v_i| \ll 1$, $|\epsilon_{si}/s_i| \ll 1$), then the system is approximately LQG balanced.

From Eqs. (25) and (27) it follows that for $Q = diag(q_i) = V = diag(v_i)$, the system is approximately LQG balanced. Indeed, the balanced CARE/FARE solution is

$$S \cong P \cong M = diag(\mu_i), \quad \mu_i = (\beta_i - 1)/2\gamma_i^2, \quad \beta_i^2 = 1 + 2q_i\gamma_i^2/\zeta_i\omega_i \tag{29}$$

Next it is shown that the weight Q

$$Q = diag(0,0,\ldots,q_i I_2,\ldots,0,0), \; q_i \leq q_{oi} \tag{30a}$$

shifts the i th pair of complex poles of the flexible structure, and leaves the remaining pairs of poles almost unchanged. only the real part of the pair of poles is changed (just moving the pole apart from the imaginary axis and stabilizing the system), and the imaginary part of the poles remains unchanged.

Result 5a. For the weight Q as in Eq. (30a) and $q_i \leq q_{oi}$, the closed-loop pair crf flexible poles $(\lambda_{cri}, \pm j\lambda_{cii})$ relates to the open loop poles $(\lambda_{ori}, \pm j\lambda_{oii})$ as follows

$$(\lambda_{cri}, \pm j\lambda_{cii}) \cong (\beta_{pi}\lambda_{ori}, \pm j\lambda_{oii}), \qquad i = 1,\ldots,n \tag{31a}$$

where β_{pi} is defined in Eq. (25). For proof see the Appendix.

The real parts of the poles are shifted by β_{pi}, while the imaginary part remains unchanged. The above proposition has additional interpretations. Note that the real part of the open-loop pole is $\lambda_{oi} = -\zeta_i\omega_i$ and that the real part of the closed-loop pole is $\lambda_{ci} = -\zeta_{ci}\omega_i$; note also that the height of the open-loop resonant peak is $\alpha_{oi} = \kappa/2\zeta_i\omega_i$, where κ is a constant, and the closed-loop resonant peak is $\alpha_{ci} = \kappa/2\zeta_{ci}\omega_i$. From (31a) one obtains $\beta_{pi} = \lambda_{cri}/\lambda_{ori}$, hence it is not difficult to see that

$$\beta_{pi} = \zeta_{ci}/\zeta_i = \alpha_{oi}/\alpha_{ci} \tag{32}$$

i.e., that β_{pi} is a ratio of closed- and open-loop damping factors, or that it is a ratio of open- and closed loop resonant peaks. Therefore, if a suppression of the i-th resonant peak by the factor β_{pi} is required, the appropriate weight q_i is determined from Eq.(25)

$$q_i = 0.5(\beta_{pi}^2 - 1)\zeta_i\omega_i\gamma_i^{-2} \tag{32}$$

Note the relatively large β_{pi}, even for small q_i, *i.e.*, a significant pole shift 10 the left. Also, β_{pi} increases with the increase of γ_i, and decreases with the increase of $\zeta_i\omega_i$, i.e., there is a significant pole shift for highly observable and controllable states with small damping.. In terms of the transfer function profile, the weight q_i suppresses the resonant peak at frequency ω_i while leaving the natural frequency unchanged. Due to weak coupling between the states, the assignment of 00C pair of stales does not significantly impact other states. Thus the weight assignment can be done for each pair of states separately.

The estimator poles are shifted in a similar manner. Denote

$$V = diag(0,0,\ldots,v_i I_2,\ldots,0,0), \; v_i \leq v_{oi}, \tag{30b}$$

then the following is true:

Result 5b. For the weight V as in Eq. (30b) and $v_i \leq v_{oi}$, the estimator pair of poles $(\lambda_{eri}, \pm j\lambda_{eii})$ relates to the open-loop poles $(\lambda_{ori}, \pm j\lambda_{oii})$ as follows

$$(\lambda_{eri}, \pm j\lambda_{eii}) \cong (\beta_{ei}\lambda_{ori}, \pm j\lambda_{oii}), \; i = 1,\ldots,n \tag{31b}$$

where β_{ei} is defined in Eq. (27). Proof is similar to Result 2a.

The limiting values q_{oi} and v_{oi} in Results 2a and 2b are determined. Their values are rather fuzzy numbers. Despite their fuzziness they arc not difficult to determine anyway. There are several symptoms that q_i is approaching q_{oi}, or that v_i is approaching v_{oi}. In the controller case, q_{oi} is the weight for which the i-th pair of complex poles of the plant departs from the horizontal trajectory in the root locus plant,, or it is the weight for which the i th resonant peak of the plant transfer function disappears (the peak is flattened). And in the estimator case, v_{oi} is the covariance for which the i-th pair of complex poles of the estimator departs from the horizontal trajectory in the root- locus plane, or ii is a covariance for which the i-th resonant peak of the estimator transfer function disappears,

5. Reduced-Order Compensator

From an implementation point of view it is crucial to obtain a compensator of the smallest possible order that preserves the stability and performance of the full-order compensator. Although the size of a plant determines the size of a compensator, in order to assure the quality of the closed-loop system, the plant model is not reduced excessively in advance. Therefore, tbc compensator reduction is a part of compensator design. The balanced LQG design procedure provides this opportunity.

In order to successfully perform the compensator reduction, an index of the importance of each compensator component is introduced. In the open-loop case Hankel singular values serve as reduction indices. In the closed loop case the characteristic values were used as reduction indices by Jonckheere and Silverman (1983). They are not a good choice, however, since they do not properly reflect the effectiveness of the compensator.

The proposed effectiveness of the closed loop system is evaluated by the degree of damping of flexible motion of the structure. The damping, on tbc other hand, depends on the pole mobility 10 the right-hand side of the complex plane. Therefore, if a particular pair of poles is easily moved (i.e., when small weight is required 10 move the poles), the respective states are easy to control and 10 estimate. On the contrary, if a particular pair of poles is difficult to move (i.e., even a large weight insignificantly moves the poles), the respective states are difficult to control and 10 estimate, In the latter case the action of the compensator is irrelevant, and the states which are difficult to control and estimate can be reduced; this demonstrates that pole mobility is a good indicator of the importance of a particular compensator state.

Consider an LQG balanced system, and denote the pole mobility index σ_i as a product of the a square of Hankel singular value and the characteristic value of a system

$$\sigma_i = \gamma_i^2\mu_i \tag{34a}$$

This combines the system observability and controllability properties of the open loop system with the compensator performance. The larger the Hankel singular value of the component, the larger the corresponding mobility index (cf. Fig.2b). Also, the more heavily weighted the component, the larger its pole mobility index (see Fig.2a). In order to show that σ_i is connected with the pole mobility, note from Eqs. (29) and (34a) that

$$\sigma_i = 0.5(\beta_i - 1) \tag{35}$$

For $\beta_i = 1$ the *i-th pole is* stationary, and σ_i is equal to zero; for a shifted pole one obtains $\beta_i > 1$ and $\sigma_i > 0$. The matrix Σ of pole mobility indices is defined

$$\Sigma = diag(\sigma_1, \sigma_2, \ldots, \sigma_{n-1}, \sigma_n) \tag{34b}$$

and from Eq.(34a) one obtains

$$\Sigma = \Gamma^2 M \tag{34c}$$

In the following, a reduction technique is discussed. Assume Σ in Eq.(34b) has a descending order, i.e., $\sigma_i \geq 0$, $\sigma_{i+1} \leq \sigma_i$, $i = 1, \ldots, n$, and divide it as follows

$$\Sigma = diag(\Sigma_r, \Sigma_t) \tag{36}$$

where Σ_r consists of first k entries of Σ, and Σ_t the remaining ones. If the entries of Σ_t are small in comparison with the entries of Σ_r, the compensator is reduced by truncating its last n-k states. Note that the value of σ_i depends on weight q_i, and if for a given weight the resonant peak is too large to be accepted (or a pair of poles too close to the imaginary axis) the weighting of this particular component should be increased to damp this particular component. The growth of weight increases the value of σ_i, which can save this particular component from reduction.

In order to investigate stability and performance of the reduced-order compensator, consider the closed-loop system as in Fig. 1. Denoting the state $x_o = [x^T \varepsilon^T]^T$, where $\varepsilon = x - \hat{x}$, one obtains the closed loop equations

$$\dot{x}_o = A_o x_o + B_o u + B_v v + B_w w, \qquad y = C_o x_o \tag{37a}$$

where

$$A_o = \begin{bmatrix} A - BK_p & BK_p \\ 0 & A - K_e C \end{bmatrix}, \quad B_o = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad B_v = \begin{bmatrix} B \\ B \end{bmatrix}, \quad B_w = \begin{bmatrix} 0 \\ -K_e \end{bmatrix}, \tag{37b}$$

$$C_o = [C \ \ -C] \tag{37c}$$

Let the matrices A, B, C be partitioned conformably to Σ in Eq.(36)

$$A = \begin{bmatrix} A_r & 0 \\ 0 & A_t \end{bmatrix}, \quad B = \begin{bmatrix} B_r \\ B_t \end{bmatrix}, \quad C = [C_r \ \ C_t] \tag{38}$$

then the reduced compensator representation is (A_r, B_r, C_r). The compensator gains are divided similarly

$$K_p = [K_{pr} \ \ K_{pt}], \quad K_e^T = [K_{er}^T \ \ K_{et}^T] \tag{39}$$

and the resulting reduced closed loop system is as follows

$$A_{or} = \begin{bmatrix} A - BK_p & BK_p \\ 0 & A_r - K_{er} C_r \end{bmatrix}, \quad B_{or} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad B_{vr} = \begin{bmatrix} B \\ B_r \end{bmatrix}, \quad B_{wr} = \begin{bmatrix} 0 \\ -K_{er} \end{bmatrix}, \tag{40a}$$

$$C_{or} = [C \ \ -C_r] \tag{40b}$$

Although (A_r, B_r, C_r) is stable, the stability of the closed-loop system with reduced compensator (A_{or}, B_{or}, C_{or}) is neither obvious nor guaranteed. But one can determine when to expect a stable closed-loop system with the reduced-order compensator.

In order to discuss this question, introduce (38) and (39) to (37b) to obtain

$$A_o = \begin{bmatrix} A_r - B_r K_{pr} & -B_r K_{pt} & B_r K_{pr} & B_r K_{pt} \\ -B_t K_{pr} & A_t - B_t K_{pt} & B_t K_{pr} & B_t K_{pt} \\ 0 & 0 & A_r - K_{er} C_r & -K_{er} C_t \\ 0 & 0 & -K_{et} C_r & A_t - K_{et} C_t \end{bmatrix} \tag{41}$$

Consider now the term $B_t K_p$

$$B_t K_p = B_t B^T S = [B_t B_r^T S_r \ \ B_t B_t^T S_t] \cong [0 \ \ diag(2\zeta_i \omega_i \sigma_i)] \tag{42}$$

where $B_t B_t^T \cong diag(2\zeta_i \omega_i \mu_i)$, $i = 1, \ldots, q$ is used. Eq.(42) shows that for small σ_i one obtains small $B_t K_p$, and in consequence small $B_t K_{pr}$ and $B_t K_{pt}$. In a similar way it can be shown that $K_e C_t$ is small. Therefore, for small σ_i the closed loop matrix as in Eq.(41) is as follows

$$A_o \cong \begin{bmatrix} A_r - B_r K_{pr} & -B_r K_{pt} & B_r K_{pr} & B_r K_{pt} \\ 0 & A_t & 0 & 0 \\ 0 & 0 & A_r - K_{er} C_r & 0 \\ 0 & 0 & -K_{et} C_r & A_t \end{bmatrix} \tag{43}$$

which shows that the poles of a truncated system have not been changed significantly, and that the poles of the retained subsystem are not influenced by the truncated part (negligible spillover). The system with the reduced compensator is stable. Of course, *since Eq.(43) represents an approximation of A_o, the* above statement is not an unconditional truth, but depends on the mobility indices. If the reduced-order compensator is obtained by reducing states with small σ_i, the reduced-order compensator *is expected* to be stable. That is, although it is not guaranteed, there is a well-founded expectation to obtain a stable reduced-order compensator.

In addition to the stability evaluation, the pole mobility indices give a good estimate of the performance of the reduced-order compensator. Namely, try truncating states with small pole mobility indices the system performance will not be deteriorated significantly. As evidence, note that for A_o as in Eq.(41) the estimation error is

$$\dot{\varepsilon}_r = (A_r - K_{er} C_r)\varepsilon_r - K_{er} C_t \varepsilon_t, \qquad \dot{\varepsilon}_t = -K_{et} C_r \varepsilon_r + (A_t - K_{et} C_t)\varepsilon_t \tag{44a}$$

and from Eq.(43) the error of the reduced-order compensator is determined

$$\dot{\varepsilon}_{rr} = (A_r - K_{er} C_r)\varepsilon_{rr}, \qquad \dot{\varepsilon}_{tt} = -K_{et} C_r \varepsilon_{rr} + A_t \varepsilon_{tt} \tag{44b}$$

It was already shown that $K_{er} C_t \cong 0$, and $K_{et} C_t \cong 0$ for small σ_i, thus $\varepsilon_r \cong \varepsilon_{rr}$ and $\varepsilon_t \cong \varepsilon_{tt}$, i.e., the estimation errors and truncation errors of the full-order and the reduced-order compensators are almost the same. Similar properties can be shown for the controller performance. The performance of full- and reduced order compensators is compared later in the application section.

As an alternative measure of performance of the closed-loop system, consider an index n_i

$$n_i = \gamma_{oi}^2 / \gamma_{ci}^2 \tag{45a}$$

It is a ratio of the open-loop Hankel singular value to the closed loop Hankel singular value, and can be also interpreted as a ratio of variances of open-loop (σ_{oi}^2) and closed loop (σ_{ci}^2) states excited by the white-noise input

$$n_i = \sigma_{oi}^2 / \sigma_{ci}^2 \tag{45b}$$

Obviously, if the *i-th* closed-loop variance is small in comparison to the *i-th* open-loop variance, the controller action at the *i-th* state is considered important, thus the state is not deleted. If the closed-loop variances are about the same as the open-loop variances, the controller action is considered marginal, and the state can be deleted without loss of per formance. In order to determine n, in a closed form the closed loop Lyapunov equation is considered

$$(A-BB^TS)\Gamma_c^2+\Gamma_c^2(A-BB^TS)^T+BB^T=0 \tag{46a}$$

or, for the i-th pair of variables

$$(A_i-B_iB_i^Ts_i)\gamma_{ci}^2+\gamma_{ci}^2(A_i-B_iB_i^Ts_i)^T+B_iB_i^T\cong 0 \tag{46b}$$

Introducing Eq.(A.3) from Appendix gives

$$\gamma_{ci}^2+2\gamma_{ci}^2\gamma_{oi}^2s_i\gamma_{oi}^2\cong 0 \tag{47}$$

or,

$$\pi_i=-\gamma_{oi}^2/\gamma_{ci}^2\cong 1+2s_i\gamma_{oi}^2=\beta_i \tag{48}$$

thus the ratio of closed- and open-loop response to white noise is equal 10 the pole shift. Another useful interpretation follows from Eq.(47)

$$\sigma_i=-0.5(\gamma_{oi}^2-\gamma_{ci}^2)/\gamma_{ci}^2 \tag{49}$$

i.e. the pole mobility index is proportional to relative change 10 while noise response of the open- and closed loop systems.

6. Applications

A simple 3-degree-of-freedom system is considered as in Fig.3, with masses $m_1=m_2=m_3=1$, stiffness $k_1=10$, $k_2=3$, $k_3=4$, and a damping matrix $D=0.004K+0.001M$, where K, M are stiffness and mass matrices, respectively. The input force is applied to the mass m_3; the output is the rate of the same mass, and the poles of the open-loop system are $\lambda_{o1,o2}=-0.0024\pm j0.9851$, $\lambda_{o3,o4}=-0.0175\pm j2.9197$, and $\lambda_{o5,o6}=-0.0295\pm j3.8084$. The weight matrix Q and the covariance matrix V are chosen as follows: $Q=V=diag(0.4, 0.4, 2, 2, 6, 6)$. The nonzero entries of Q and V shift the poles to the right, so that the peaks in the closed-loop transfer function are flattened as in Fig.4. The matrix V is chosen to be equal to Q to obtain a balanced LQG compensator. For these matrices the solution S of CARE and the solution P of FARE are equal and diagonally dominant,

$$S=P\cong M=diag(1.3288, 1.3261, 4.3161, 4.1301, 25.2817, 24.01\$0),$$

and the corresponding gains are sign-symmetric (Jonckheere and Silverman, 1983)

$$k_p=[0.0039, 0.889.3, 0.2978, -1.9291, 2.1378, -2.1636]$$

$$k_e^T=[-0.0039, 0.8893, -0.2978, -1.9291, -2.1378, -2.1636]$$

The Hankel matrix of the plant is

$$\Gamma=diag(7.9776, 7.9776, 2.2.337, 2.3336, 0.4893, 0.48\$0)$$

thus the matrix Σ is obtained

$$\Sigma\cong diag(84.5658, 84.3920, 21..5332, 20.6058, 6.0153, 5.7586)$$

Poles of the open-loop plant, closed-loop systems and estimator are shown in Fig.5. The closed-loop poles and estimator poles were shifted horizontally with respect to the open-loop poles! in agreement with the Results 2a and 2b. For the chosen weights the projected (from Eq.(32)) and actual shifts are 137 vs 146 for the first pair of poles, 33 vs 34 for the second pair of poles, and 8.5 vs 10 for the third pair of poles. Moreover, since the copensator is balanced, the poles of the closed loop system and the estimator overlap. 'the. closed-loop impulse response in Fig.6 (solid line), shows good vibration damping properties, which is also confirmed try the closed-hop transfer function, Fig.4 (dashed line). The compensator is reduced from six to four state variables. The truncated states are related to the smallest diagonal entries (5.7586, 6.0453) of Σ. The impulse response of the full and reduced-order compensator are compared in Fig.6, showing good coincidence. However, if the two states corresponding to the medium values of Σ (20.6058, 21.5.332) arc. deleted, the performance of the reduced-order compensator is significantly deteriorated, and if the slates corresponding to the largest entries of Σ are reduced, the compensator is unstable.

Next, the application of the LQG compensator to the truss structure from Fig.7 is investigated. For this structure $l_1=70$ in., $l_2=100$ in., each truss has a cross-section area of 2 in.², elastic modulus of 10^6 lb/in.², and mass density of 2 lb sec²/in.². Vertical control forces are applied at nodes $na1$ and $na2$, and the output roles are measured in the vertical direction at nodes $no1$ and $no2$. The system has 26 states (13 balanced components), two inputs, and two outputs. The weight (Q) and covariance (V) matrices are assured equal and diagonal, $Q=V=diag(q_1,q_1,q_2,q_2,q_3,...,q_{13},q_{13})$, where $q_1=200$, $q_2=100$, $q_3=1000$, $q_4=2000$, $q_5=10000$, $q_6=20000$, $q_7=....q_{13}=200$. The CARE and FARE solutions are diagonally dominant, so that the resulting matrix Σ is diagonally dominant. The plots of diagonal entries of Σ are plotted in Fig.8. Poles of the open-loop structure as well the closed-loop system and the estimator are shown in Fig.9. For the balanced compensator the pole.s of the closed-loop system and the estimator overlap. The open-loop transfer functions arc shown in Fig. 10 (solid line) and the closed loop transfer functions of the plant and estimator overlap in Fig. 10 (dashed line), and show that the Oscillatory motion of the structure is damped out. The compensator is reduced by truncating the 14 states, which correspond 10 the small pole mobility indices 0,<0.5. The resulting reduced-order compensator has 12 states, The impulse responses of the full and reduced-order compensator are compared in Fig. 11, showing good coincidence.

7. Conclusions

The properties of Moore balanced representation of flexible structures have been extended for the closet-loop systems. A balanced LQG compensator is obtained that pays the same attention to controlling and to estimating the system. The properties of the balanced LQG system arc used to obtain a rcduccd-order compensator that preserves the stability and performance of the full-order compensator, as illustrated with the LQG balanced control of a truss structure. Since LQG balancing and Moore open-foop balancing coincide for flexible structures, the open-loop reduction (based on Moore balancing) and the LQG reduction form a unified approach to system reduction, useful duc to its simplicity.

Acknowledgment

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Appendix. Proofs

Proof of Result 4a. For a flexible system with n components (or $N=2n$ states), the balanced grammian has the following form (see Gawronski and Juang, 1990; Gawronski and Williams, 1991)

$$\Gamma \cong diag(\gamma_1,\gamma_1,\gamma_2,\gamma_2,\dots,\gamma_n,\gamma_n), \tag{A.1}$$

and the matrix A is almost block-diagonal, with dominant 2x2 blocks on the main diagonal

$$A \cong diag(A_i), \qquad A_i = \begin{bmatrix} -\zeta_i\omega_i & -\omega_i \\ \omega_i & -\zeta_i\omega_i \end{bmatrix}, \quad i=1,\dots,n \tag{A.2}$$

where ω_i is the i-th natural frequency, and ζ_i is the i-th modal damping. Introducing Eqs.(A.1) and (A.2) to (1) gives

$$\gamma_i^2(A_i + A_i^T) \cong -B_iB_i^T \cong -C_i^TC_i. \tag{A.3}$$

Due to diagonally dominant matrix A for a flexible structure in balanced representation, and for Q as in Eq.(24), there exist $q_i \le q_{oi}$, $i=1,\dots,n$, such that the solution S of the Riccati equation (6) is also diagonally dominant with 2×2 blocks S_i on the main diagonal

$$S_i \cong s_iI_2, \quad s_i > 0, \qquad i=1,\dots,n. \tag{A.4}$$

Thus, equation (6) turns into a set of the following equations

$$s_i(A_i + A_i^T) - s_i^2B_iB_i^T + q_iI_2 = 0, \qquad i=1,\dots,n. \tag{A.5}$$

For a balanced system $B_iB_i^T \cong -\gamma_i^2(A_i + A_i^T)$, see Eq. (A.3), and $A_i + A_i^T = -2\zeta_i\omega_iI_2$, see Eq. (A.2). Therefore Eq. (A.5) is now

$$s_i^2 + s_i/\gamma_i^2 - 0.5q_i/\zeta_i\omega_i\gamma_i^2 = 0, \qquad i=1,\dots,n. \tag{A.6}$$

There are two solutions of Eq. (A.6), but for a stable system and for $q_i = 0$ it is required that $s_i = 0$, therefore (25) is the unique solution of Eq. (A.6).

Proof of Result 5a. For small q_i the matrix A of the closed-loop system is diagonally dominant $A_o \cong diag(A_{oi})$, $i=1,\dots,n$, and $A_{oi} = A_i - B_iB_i^Ts_i$, introducing Eq.(A.3) one obtains

$$A_{oi} \cong A_i + 2s_i\gamma_i^2(A_i + A_i^T), \tag{A.7}$$

and introducing A_i as in Eq.(A.2) to Eq.(A.7) one obtains

$$A_{oi} = \begin{bmatrix} -\beta_{pi}\zeta_i\omega_i & -\omega_i \\ \omega_i & -\beta_{pi}\zeta_i\omega_i \end{bmatrix}, \tag{A.8}$$

with β_{pi} as in Eq. (25).

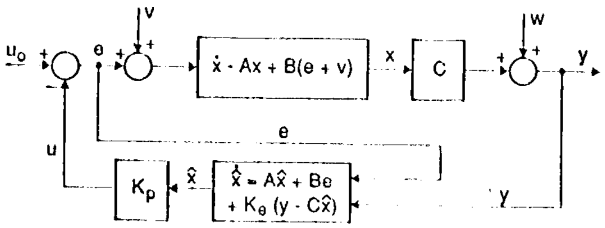

Fig. 1, Block diagram of flex. structure with LQG compensator.

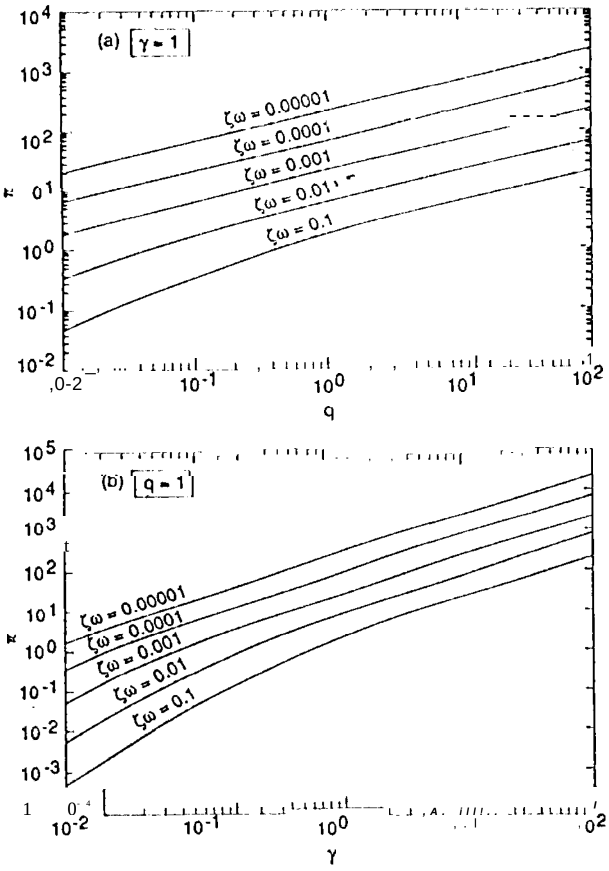

Fig.2. Pole mobility index vs. weight.

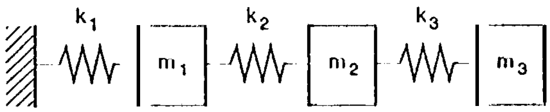

Fig.3. Simple flexible system.

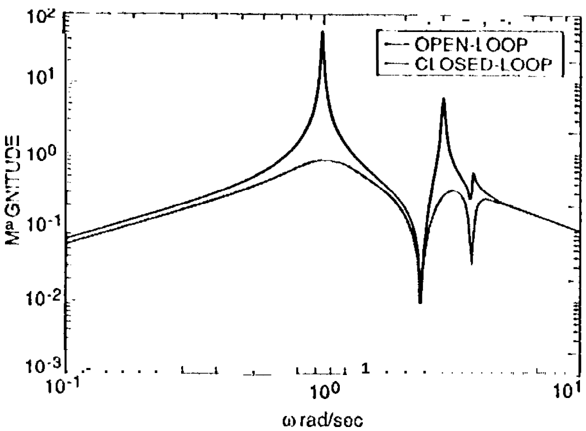

Fig.4. Transfer functions of simple flexible system.

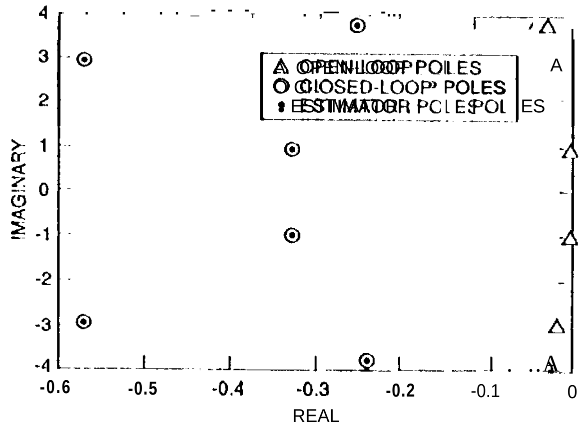

Fig.5. Poles of simple flexible system, and estimator.

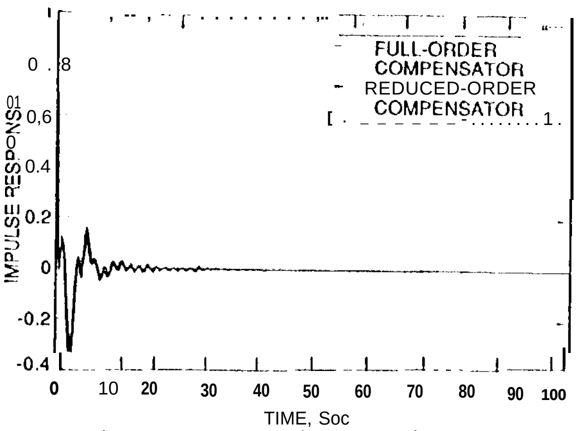

Fig.6. Impulse responses of full and reduced compensator.

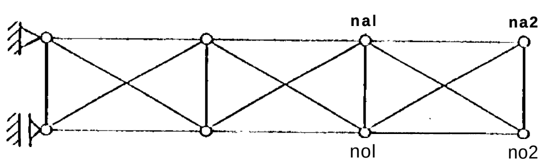

Fig.7. Truss structure.

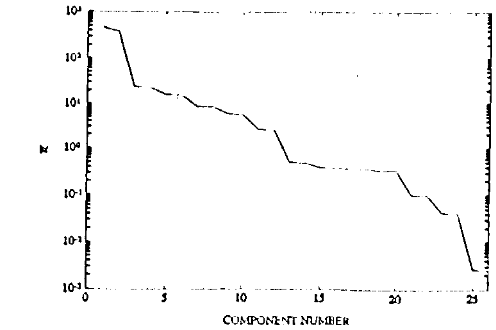

Fig.8 Pole mobility indices for tress structure.

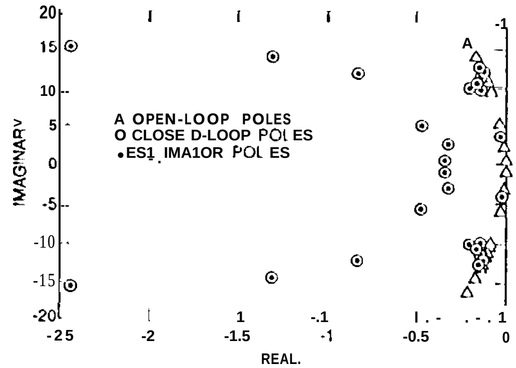

Fig.9. Poles of tress structure, and estimator.

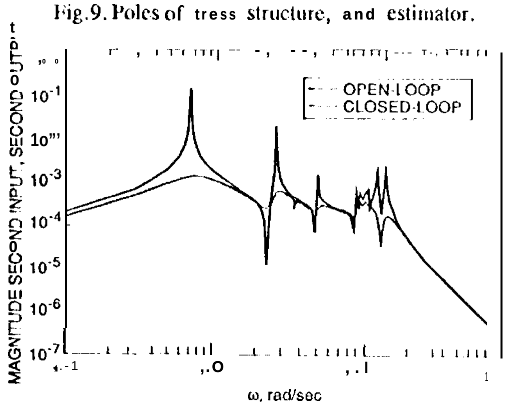

Fig. 10. Transfer functions of truss structure.

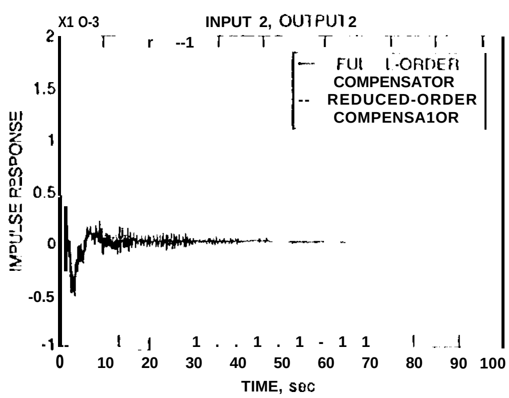

Fig. 11. Impulse responses of full and reduced compensator.